Core 4 - June 2013

1) a) $\dfrac{5-8x}{(2+x)(1-3x)} = \dfrac{A}{2+x} + \dfrac{B}{1-3x}$

$\rightarrow \quad 5-8x = A(1-3x) + B(2+x)$

Let $x = 1/3$ $\quad 5 - 8/3 = B(2 + 1/3)$

$\rightarrow \quad 7/3 = B(7/3) \quad \rightarrow \quad B = 1$

Let $x = -2$ $\quad 5 - 8(-2) = A(1 - 3(-2))$

$\rightarrow \quad 21 = A(7) \quad \rightarrow \quad A = 3$

$\rightarrow \quad \dfrac{3}{2+x} + \dfrac{1}{1-3x}$

ii) $\displaystyle\int_{-1}^{0} \frac{3}{2+x} + \frac{1}{1-3x} = \left[3\ln(2+x) - \tfrac{1}{3}\ln(1-3x)\right]_{-1}^{0}$

$= \left[3\ln(2) - \tfrac{1}{3}\ln(1)\right] - \left[3\ln(1) - \tfrac{1}{3}\ln(4)\right]$

$= 3\ln(2) + \tfrac{1}{3}\ln(4)$

$= 3\ln(2) + \tfrac{1}{3}\ln(2^2)$

$= 3\ln(2) + \tfrac{2}{3}\ln(2)$

$= 11/3 \ln(2)$

b) i) $6x^2 \div 3x^2 = 2 \quad \rightarrow \quad c = 2$

ii) Area $= \displaystyle\int_{-1}^{0} 2 + \frac{5-8x}{2-5x-3x^2}$

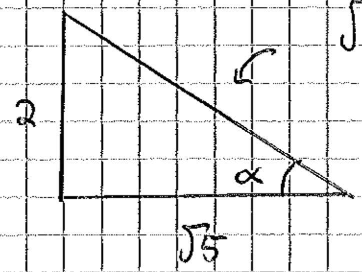
$= \displaystyle\int_{-1}^{0} 2 + \int_{-1}^{0} \frac{5-8x}{(2+x)(1-3x)}$

$= \left[2x\right]_{-1}^{0} + 11/3 \ln(2)$

$= -(-2) + 11/3 \ln(2)$

$= 2 + 11/3 \ln(2)$

2) a) i)



$\sqrt{2^2 + (\sqrt{5})^2} = \sqrt{9} = 3$

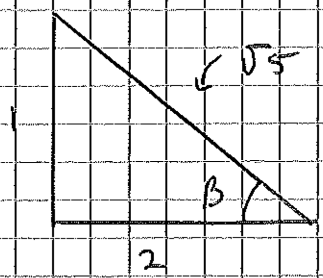
$\rightarrow \quad \sin\alpha = 2/3$

$\rightarrow \quad \cos\alpha = \sqrt{5}/3$

ii) $\sin 2\alpha = 2\sin\alpha\cos\alpha$

$$= 2\left(\tfrac{2}{3}\right)\left(\tfrac{\sqrt{5}}{3}\right) = \tfrac{4}{9}\sqrt{5}$$

b)



$\sin\beta = 1/\sqrt{5}$

$\cos\beta = 2/\sqrt{5}$

$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

$$= \left(\sqrt{5}/3\right)\left(2/\sqrt{5}\right) + \left(2/3\right)\left(1/\sqrt{5}\right)$$

$$= 2\sqrt{5}/3\sqrt{5} + 2/3\sqrt{5}$$

↓ rationalise denominator

$$= 2/3 + 2\sqrt{5}/15$$

$$= 2/15\left(5 + \sqrt{5}\right)$$

③ a) $$(1+6x)^{-1/3} = 1 + \left(-\tfrac{1}{3}\right)(6x) + \frac{\left(-\tfrac{1}{3}\right)\left(-\tfrac{4}{3}\right)}{2!}(6x)^2$$

$$= 1 - 2x + 8x^2$$

b) i) $(27+6x)^{-1/3}$

$$= 27^{-1/3}\left(1 + 6/27x\right)^{-1/3}$$

$$= 1/3\left[1 - 2/27\,x + 8/27^2\,x^2\right]$$

$$= 1/3\left[1 - 2/27\,x + 8/729\,x^2\right]$$

$$= 1/3 - 2/81\,x + 8/2187\,x^2$$

ii) $$\frac{2}{\sqrt[3]{28}} = 2\left[28\right]^{-1/3} = 2\left[27+1\right]^{-1/3}$$

Let $x = 1/6 \rightarrow 2\left[27 + 6(1/6)\right]^{-1/3}$

$$= 2\left[1/3 - 2/81\,(1/6) + 8/2187\,(1/6)^2\right]$$

$$= 0.6586639\ldots \quad (6dp)$$

④ a) $\dfrac{dx}{dt} = -16e^{-2t}$ $\qquad$ $\dfrac{dy}{dt} = 4e^{2t}$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = \cancel{-16e^{-2t} \times 4e^{2t}}$$

$$4e^{2t} \times \frac{1}{-16e^{-2t}}$$

$$= -1/4\,e^{4t}$$

b) i) $t = \ln(2) \rightarrow dy/dx = -1/4\, e^{4(\ln 2)}$

$= -1/4\, e^{\ln(16)} = -1/4 \times 16 = -4$

ii) $x = 8e^{-2(\ln 2)} - 4 = 8e^{\ln 1/4} - 4 = 2 - 4 = -2$

$y = 2e^{2(\ln 2)} + 4 = 2e^{\ln(4)} + 4 = 8 + 4 = 12$

$\therefore$ co-ordinates are $(-2, 12)$

iii) Gradient of normal $= 1/4$

$x_1 = -2$

$y_1 = 12$

Equation: $y - 12 = 1/4(x + 2)$

crosses x-axis when $y = 0$

$\rightarrow -12 = 1/4(x + 2)$

$\rightarrow -48 = x + 2$

$\rightarrow x = -50 \quad (-50, 0)$

c) $xy = (8e^{-2t} - 4)(2e^{2t} + 4)$

$= 16 + 32e^{-2t} - 8e^{2t} - 16 = 32e^{-2t} - 8e^{2t}$

$4y = 4(2e^{2t} + 4) = 8e^{2t} + 16$

$-4x = -4(8e^{-2t} - 4) = -32e^{-2t} + 16$

$\therefore xy + 4y - 4x = 32$

5) a) Need $2x + 3 = 0 \rightarrow x = -3/2$

$f(-3/2) = 4(-3/2)^3 - 11(-3/2) - 3$

$= 4(-27/8) + 33/2 - 3 = 0$

$\therefore (2x + 3)$ is a factor

b)

$$\begin{array}{r|l} & 2x^2 - 3x + 1 \\ \hline 2x+3 & 4x^3 + 0x^2 - 11x - 3 \\ & 4x^3 + 6x^2 \\ & \quad -6x^2 - 9x \\ & \qquad -2x - 3 \end{array}$$

Algebraic Long Division

$\rightarrow (2x + 3)(2x^2 - 3x + 1)$

c) i) $2\cos(2\theta)\sin\theta + 9\sin\theta + 3 = 0$

$= 2(1 - 2\sin^2\theta)\sin\theta + 9\sin\theta + 3 = 0$

$= 2\sin\theta - 4\sin^3\theta + 9\sin\theta + 3 = 0$

$\rightarrow 4\sin^3\theta - 11\sin\theta - 3 = 0$

$\rightarrow 4x^3 - 11x - 3 = 0 \quad [x = \sin\theta]$

ii) $(2x + 3)(2x^2 - 3x - 1) = 0$

$x = -3/2$

$\sin\theta = -3/2$

NO SOLUTIONS

$2x^2 - 3x - 1 = 0$

Formula: $x = \dfrac{3 \pm \sqrt{9 - 4 \times 2 \times -1}}{4}$

$\rightarrow x = \dfrac{3 \pm \sqrt{17}}{4}$

$x = 1.780...$ or $x = -0.2807...$

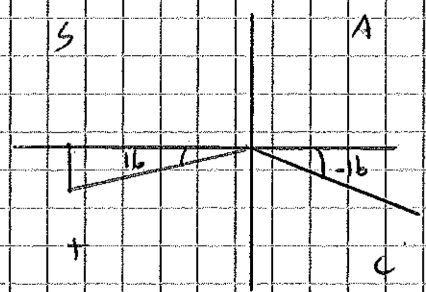
$\sin\theta = 1.780$

$\rightarrow$ NO SOLUTIONS

$\sin\theta = -0.2807...$

$\theta = \sin^{-1}(-0.2807)$

$= -16.3^\circ$



$\theta = 344^\circ$

and

196°

(6) a) $\begin{bmatrix} 3 \\ -2 \\ 4 \end{bmatrix} + \lambda \begin{bmatrix} 7 \\ -7 \\ 5 \end{bmatrix} = \begin{bmatrix} -4 \\ 5 \\ -1 \end{bmatrix} \rightarrow \begin{aligned} 3 + 7\lambda &= -4 \\ -2 - 7\lambda &= 5 \\ 4 + 5\lambda &= -1 \end{aligned}$

All satisfied by $\lambda = -1$

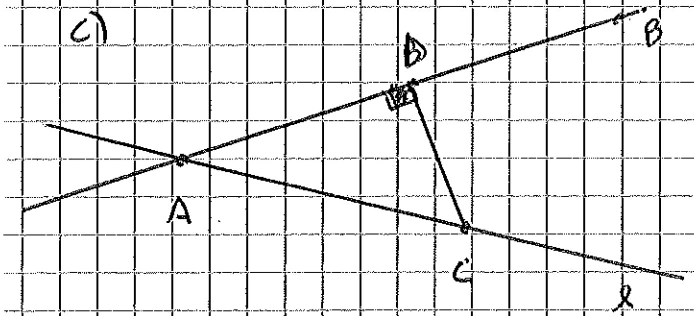
$\therefore C$ lies on line

b) $\vec{AB} = \vec{AO} + \vec{OB}$

$= \begin{bmatrix} -3 \\ 2 \\ -4 \end{bmatrix} + \begin{bmatrix} 1 \\ -5 \\ 6 \end{bmatrix} = \begin{bmatrix} -2 \\ -3 \\ 2 \end{bmatrix}$

$\therefore$ Equation $= \begin{bmatrix} 3 \\ -2 \\ 4 \end{bmatrix} + \mu \begin{bmatrix} -2 \\ -3 \\ 2 \end{bmatrix}$

c)



As D lies on AB,

its co-ordinates must be:

$$\begin{pmatrix} 3-2\mu \\ -2-3\mu \\ 4+2\mu \end{pmatrix}$$

$$\vec{CD} = \vec{CO} + \vec{OD}$$

$$= \begin{bmatrix} 4 \\ -5 \\ 1 \end{bmatrix} + \begin{pmatrix} 3-2\mu \\ -2-3\mu \\ 4+2\mu \end{pmatrix} = \begin{bmatrix} 7-2\mu \\ -7-3\mu \\ 5+2\mu \end{bmatrix}$$

As 90°, $\vec{CD} \cdot \vec{AB} = 0$

$$\rightarrow \begin{pmatrix} 7-2\mu \\ -7-3\mu \\ 5+2\mu \end{pmatrix} \cdot \begin{pmatrix} -2 \\ -3 \\ 2 \end{pmatrix} = 0$$

$$\rightarrow -14 + 4\mu + 21 + 9\mu + 10 + 4\mu = 0$$

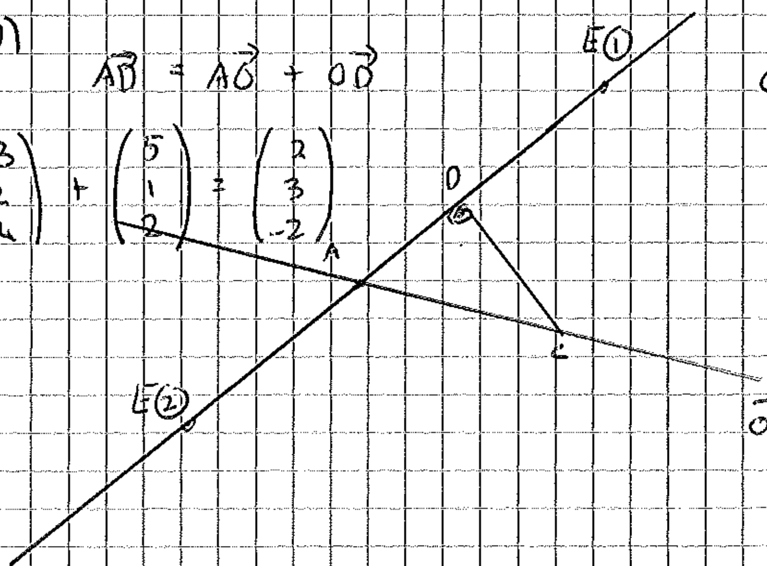
$$\rightarrow 17\mu + 17 = 0 \rightarrow \mu = -1$$

$\therefore$ co-ordinates of D are:

$$3 - 2(-1) = 5$$
$$-2 - 3(-1) = 1$$
$$4 + 2(-1) = 2$$

$$\rightarrow (5, 1, 2)$$

d)



$$\vec{AD} = \vec{AO} + \vec{OD}$$

$$= \begin{pmatrix} -3 \\ 2 \\ -4 \end{pmatrix} + \begin{pmatrix} 5 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ -2 \end{pmatrix}$$

$$\vec{OE_1} = \vec{OA} + 3\vec{AD}$$

$$= \begin{pmatrix} 3 \\ -2 \\ 4 \end{pmatrix} + 3\begin{pmatrix} 2 \\ 3 \\ -2 \end{pmatrix} = \begin{pmatrix} 9 \\ -7 \\ -2 \end{pmatrix}$$

$$\vec{OE_2} = \vec{OA} - 3\vec{AD}$$

$$= \begin{pmatrix} 3 \\ -2 \\ 4 \end{pmatrix} - 3\begin{pmatrix} 2 \\ 3 \\ -2 \end{pmatrix} = \begin{pmatrix} -3 \\ -11 \\ 10 \end{pmatrix}$$

$\therefore$ Possible co-ordinates of E

$= (9, -7, -2)$ or $(-3, -11, 10)$

(7) Max value $= 1.3 \rightarrow a = 1.3$

cos repeats every 2π, $\therefore k$ must be $2\pi/12$

$$\rightarrow \frac{dh}{dt} = 1.3\cos\left(\frac{2\pi}{12}t\right)$$

(8) a) $\int t\cos\left(\frac{\pi}{4}t\right)dt$

Integration by parts $\quad u = t \quad \frac{dv}{dt} = \cos\left(\frac{\pi}{4}t\right)$

$\frac{du}{dt} = 1 \quad v = \frac{4}{\pi}\sin\left(\frac{\pi}{4}t\right)$

$$\rightarrow uv - \int v\frac{du}{dt}$$

$$\rightarrow t\frac{4}{\pi}\sin\left(\frac{\pi}{4}t\right) - \int \frac{4}{\pi}\sin\left(\frac{\pi}{4}t\right)$$

$$\rightarrow t\frac{4}{\pi}\sin\left(\frac{\pi}{4}t\right) + \frac{4}{\pi}\times\frac{4}{\pi}\cos\left(\frac{\pi}{4}t\right)$$

$$\rightarrow t\frac{4}{\pi}\sin\left(\frac{\pi}{4}t\right) + \frac{16}{\pi^2}\cos\left(\frac{\pi}{4}t\right)$$

b) $\frac{dx}{dt} = \frac{t\cos\left(\frac{\pi}{4}t\right)}{32x}$

$$\int 32x\,dx = \int t\cos\left(\frac{\pi}{4}t\right)dt$$

$$16x^2 = t\frac{4}{\pi}\sin\left(\frac{\pi}{4}t\right) + \frac{16}{\pi^2}\cos\left(\frac{\pi}{4}t\right) + c$$

when $t = 0$, $x = 4$

$$\rightarrow 16(4^2) = \frac{16}{\pi^2}\cos(0) + c$$

$$\rightarrow 256 = \frac{16}{\pi^2} + c$$

$$\rightarrow c = 256 - \frac{16}{\pi^2}$$

when $t = 45$

$$\rightarrow 16x^2 = 45\times\frac{4}{\pi}\sin\left(\frac{\pi}{4}\times 45\right) + \frac{16}{\pi^2}\cos\left(\frac{\pi}{4}\times 45\right) + 256 - \frac{16}{\pi^2}$$

$$\rightarrow 16x^2 = 212.718...$$

$$\rightarrow x^2 = 13.294...$$

$$\rightarrow x = 3.646... \text{ m}$$

$$= 3.65 \text{ m} \quad (\text{nearest cm})$$